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The mechanical quadrature method for linear and nonlinear Volterra's integral equations of the second kind.

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ABSTRACT

This paper studies the numerical evaluation of volterra's integral equations of the second kind. A deliberate attempt is made to deviate from the traditional method of representing the unknown solution by means of resolvent. This is actualized with the aid of the mechanical quadrature method. The problem is solved in Banach space setting. Some numerical results are obtained for the cases of linear and nonlinear equations.

Keywords: Volterra's integral equations, Mechanical quadrature.

INTRODUCTION

Like in many different problems of applied nature, the exact analytical solution to volterra's integral equations of the second kind.

$$y(x) - \int_a^x K(x,s)\,y(s)\,ds = f(x), \quad x \in [a,b], \qquad \dots (1.1)$$

often favours a limited class of equations, namely, the class with linear kernels. This emanates from the fact that a typical classical analytical approach consists in representing the unknown solution by means of resolvent (Volterra , 1982).

Among the most effective approximate analytical approach includes the Picards approximate method (Myskis, 1971) or any other improved interactive setting (Golavash and Kalaida, 1974). Unfortunately, they are collectively restrictive in applications in that, on the very bases of their construction lies the representation of the unknown solutions in the form of an infinite power series (Badalov, 1980).

In this paper, we are suggesting the use of mechanical quadrature method for the approximate solutions of not only the linear volterra's integral equations of the second kind, but also for the actualization of its nonlinear counterpart (Baker and Keech, 1978; Bialecki et al, 2004). The method is developed in a Banach space B (this means that the volterra operator maps a function φ from a Banach space into itself). However, to keep the treatment in a simpler level, we may, if necessary, employ the techniques of Hilbert spaces. The numerical solution is sought in the form of series

$$\tilde{y} = \sum_{i=1}^{m} c_i \varphi_i \qquad \dots (1.2)$$

where: φ_i is a system of linearly independent coordinate elements and c_i, are the unknown coefficients. To evaluate the unknown coefficie-nts, a finite system of algebraic equations is obtained. The stability of the process is demonstrated in the solved examples.

THE METHOD AND SPACE

We re-write equation (1.1) in an operator form

$$y = Ty + f \qquad \dots (2.1)$$

where *T* may be linear or nonlinear and so also may be equation (2.1). In future, we shall assume that operator *T* is linear and acts in Banach spaces. The element f is presumably known, whereas y is called the unknown function and both of them belong to some metric spaces F and Y respectively.

In the event that operator *T* is completely continuous and what is more, the number 1 is not the spectrum point of operator *T*, the inverse $U = (I - T)^{-1}$ exists and is bounded. That is, $\|U\| < \infty$ and consequently possesses the correctness property relative to the right-hand side expressible by the inequality

$$\|y - \tilde{y}\| \le \|U\| \cdot \|f - \tilde{f}\|. \qquad \dots (2.2)$$

where $\tilde{y}$ and $\tilde{f}$ are the approximating elements to y and f respectively. Notice here that the completely continuous property of *T* permits the validity of the inequality $\|y - \tilde{y}\| < \varepsilon$ whenever $\|f - \tilde{f}\| < \delta$ holds for arbitrary small positive numbers ε and δ.

We recall that if a linear operator *T* is completely continuous (co-mpact) it can be approximated, to arbitrary accuracy, with a finite-dimensional operator $\tilde{T}$ (Baker and Keech, 1978). Consequently, in place of equation (2.1), we write a rather very close equation

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$$\tilde{y} = \tilde{T}\,\tilde{y} + f. \qquad \dots\ (2.3)$$

Assuming the existence and boundedness of operator $(I - \tilde{T})^{-1}$ and the fulfillment of the condition

$$\left\|T - \tilde{T}\right\| \cdot \left\|\left(I - \tilde{T}\right)^{-1}\right\| < 1,$$

which is always possible provided 1 is not a spectrum point of $\tilde{T}$ (Yoshida, 1971), we obtain the estimate

$$\|y - \tilde{y}\| \le \frac{\left\|\left(I - \tilde{T}\right)^{-1}\right\| \cdot \|f\| \cdot \left\|T - \tilde{T}\right\|}{I - \left\|T - \tilde{T}\right\| \cdot \left\|\left(I - T\right)^{-1}\right\|}. \qquad \dots\ (2.4)$$

This guarantees the fact that the higher the degree of approximation of the operator T to $\tilde{T}$, the lesser the value of the approximating error. The accuracy of the approximating operator manifested by the indicator $\left\|T - \tilde{T}\right\|$, may be improved in two ways: increase the dimension of operator $\tilde{T}$ or, for practical purposes, choose the family of approximate operator equations which provide a greater accuracy for the same amount of calculations.

Like many approximate formulas, the proposed method consist in representing the approximate solutions of y in equation (2.1) in the form of the series

$$\tilde{y} = \sum_{i=1}^{m} c_i \varphi_i \qquad \dots\ (2.5)$$

where φ_i, $i = \overline{1, m}$ is a system of linearly independent coordinate elements contained in Banach space B. The unknown coefficients may be determined in many different ways as suggested by the requirement posed by the source error

$$\varepsilon_m = \sum_{i=1}^{m} c_i \left(\varphi_i - T\varphi_i\right) - f \qquad \dots\ (2.6)$$

To further explain this remark, we note that in analyzing the given equation in Hilbert space, the minimum value of the source error is obtained with the assistance of the least square method and the value to be minimized is given by the equation

$$\|\varepsilon_m\|^2 = \left\|\sum_{i=1}^{m} c_i \left(\varphi_i - T\varphi_i\right) - f\right\|^2. \qquad \dots\ (2.7)$$

Notice that in the above simplification, the terms involving the products of φ_I and φ_j dropped out because of the pair orthogonality property of the bases in the Hilbert space.

Differentiating both sides of (2.7) with respect to c_i, we obtain a system of linear algebraic equations

$$\sum_{i=1}^{m} c_i \left(\varphi_i - T\varphi_i\right)\left(\varphi_j - T\varphi_j\right) = f\left(\varphi_j - T\varphi_j\right) \qquad \dots\ (2.8)$$

for the determination of c_i. Subtracting the approximate solution

$$\tilde{y} = T\,\tilde{y} + f + \varepsilon_m$$

from (2.1), we obtain the estimate

$$\|y - \tilde{y}\| \le \left\|\left(I - T\right)^{-1}\right\| \cdot \|\varepsilon_m\|.$$

Finally, we conclude this section by noting that if operator $(I - T)^{-1}$ is bounded, the reduction in size of the approximate error is achieved whenever the numbers of the coordinate elements in (2.5) are increased.

LINEAR ALGORITHM

To actualize the solution of the linear equation

$$y(x) - \int_a^x K(x,s)\,y(s)\,ds = f(x), \quad x \in [a,b) \qquad \dots\ (3.1)$$

with the aid of the mechanical quadrature method it is necessary to take advantage of the expression

$$y(x_i) - \int_a^{x_i} K(x_i, s)\,y(s)\,ds = f(x_i), \quad i = \overline{1, n}, \qquad \dots\ (3.2)$$

obtained from the original equation by fixing the value of the argument *x*. The value of x_i may be chosen by a special formula or defined before hand if, for example, *f(x)* is defined in a tabular form. Taking the values of x_i as the nodal points of the quadrature formula, we replace equation (3.2) with a finite system

$$y(x_i) - \sum_{j=1}^{i} T_j k(x_i, x_j)\,y(x_j) = f(x\ ) + R_i[y], \qquad \dots\ (3.3)$$

where $R_i[y]$ is the approximating error. The validity of equation (3.3) is assured by the assumption of continuity of the kernel and the right-hand side *f(x)* in a given triangle and interval respectively. Assuming that the error $R_i[y]$ is insignificant and can therefore be ignored, we obtain a system of linear algebraic equations

$$y_i - \sum_{j=1}^{i} T_j K_{ij} y_j = f_i, \quad i = \overline{1, n}. \qquad \dots\ (3.4)$$

Here, we introduce the following symbols and will maintain same in all our future discussions:

$$\tilde{y}(x_i) = y_i, \quad f(x_i) = f_i, \quad K(x_i, x_j) = K_{ij}.$$

The solution of (3.4) provides the approximate values of the unknown function $\tilde{y}(x_i) = y_i$ at the nodal points x_i. System (3.4) may be transformed into the form

$$-\sum_{j=1}^{i-1} T_j K_{ij} y_i + \left(I - T_i K_{ii}\right) y_i = f_i, \quad i = \overline{1, n}, \qquad \dots\ (3.5)$$

or

$$\begin{bmatrix} I - T_1K_{11} & & & & & \\ -T_1K_{21} & I - T_2K_{22} & & & & \\ \dots & \dots & \dots & & \dots & \dots \\ -T_1K_{i1} & -T_2K_{i2} & & I - T_iK_{ii} & & \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -T_1K_{n1} & -T_2K_{n2} & & -T_iK_{ni} & & I - T_nK_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_i \\ \dots \\ y_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \dots \\ f_i \\ \dots \\ f_n \end{bmatrix}$$

from where it is clear that the matrix of the coefficients of the algebraic system is triangular. This allows us to successively determine $y_1, y_2, \ldots, y_n$ with the aid of the recurrent formula.

$$y_i = (I - T_i K_{ii})^{-1}\left(f_n + \sum_{j=1}^{i-1} T_j K_{ij} y_j\right), \quad i = \overline{1, n} \quad \ldots (3.7)$$

on condition that

$$(I - T_i K_{ii}) \neq O. \quad \ldots (3.8)$$

Notice that condition (3.8) is always possible with a careful choice of the nodal points and the assurance of the smallness of the coefficients T_i.

In a circumstance where the trapezoidal formula is being used, equation (3.7) takes the form

$$y_i = \left(1 - \frac{h}{2} K_{ii}\right)^{-1}\left(f_i + \frac{h}{2} K_{i1} y_1 + h \sum_{j=2}^{i-1} K_{ij} y_j\right). \ldots (3.9)$$

NONLINEAR ALGORITHM

The application of the quadrature formula to nonlinear equation

$$y(x) - \int_a^x K(x, s, y(x))ds = f(x), \quad x \in [a, b) \quad \ldots (4.1)$$

reduces it to the expression

$$y(x_i) - \int_a^{x_i} K(x_i, s, y(s))ds = f(x_i) \quad \ldots (4.2)$$

which is further transformed to a system of nonlinear recurrent formula

$$y_i - \sum_{j=1}^{i} T_j K_{ij}(y_i) = f_i, \quad i = \overline{1, n}. \quad \ldots (4.3)$$

Here, we have made use of symbol $K_{ij}(y_j)$ for $K(x_i, x_j, \tilde{y}(x_j))$. If however, the kernel K is linear in the nonlinear setting

$$y(x) - \int_a^x K(x, s) F(s, y(s))ds = f(x), \quad x \in [a, b), \ldots (4.4)$$

equation (4.2) becomes

$$y(x_i) - \int_a^{x_i} K(x_i, s) F(y(s))ds = f(x_i) \quad \ldots (4.5)$$

and the associated system of algebraic equations takes the form

$$y_i - \sum_{j=1}^{i} T_i K_{ij} F(y_i) = f(x_i)$$

where $F_j(y_j) = F(x_j, \tilde{y}(x_j))$

Equation (4.3) allows us to determine the approximate solutions y_i by way of successive evaluation of n nonlinear equations

$$y_i - T_i K_{ii}(y_i) = f_i + \sum_{j=1}^{i-1} T_j K_{ij}(y_j). \quad \ldots (4.6)$$

APPENDIX

Linear equation

We experiment the algorithm of the method for the determination of the solution to the equation

$$y(x) = 2\int_0^x e^{x^2 - t^2} y(t)dt + e^{x^2 + 2x}, \quad x \in [0, 0.1] \ldots (5.1.1)$$

where the points $x = x_i$ = 0.00; 0.02; 0.04; 0.06; 0.08; 0.10. For this purpose, we immediately take advantage of the generalized trapezoidal formula for the transformation of an integral to a finite sum given in equation (3.9). This leads to the successive determination of the approximate solutions y_i, for 6 nodal points with the step h translated to 0.02. The approximate results are as follows:

y(0.00) = y_1 = 1.000000
y(0.02) = y_2 = 1.061238
y(0.04) = y_3 = 1.147518
y(0.06) = y_4 = 1.252595
y(0.04) = y_5 = 1.353414
y(0.04) = y_6 = 1.461631

Nonlinear equation

To complete the verification of our method, we examine the nonlinear equation

$$y(x) = 2\int_0^x e^{x^2 - t^2} (y(t))^2 dt + e^{x^2 + 2x}, \quad x \in [0; 0.1], \ldots (5.2.1)$$

where $K(x_i, x_j) = 2e^{x_i^2 - x_j^2}$, $f(x) = e^{x^2 + 2x}$. Again, from equation (4.6), we quickly obtain the formula for the determination of the approximate solutions as follows:

$$y_1 = f_1, \quad y_i - 0.01 K_{ii}(y_i^2) = f_i + \sum_{j=1}^{i-1} 0.02 K_{ij}(y_j^2), \quad i = \overline{2, 6}.$$

As in appendix 5.1 above, the approximate results are as follows:

y(0.00) = y_1 = 1.00000
y(0.02) = y_2 = 1.10570
y(0.04) = y_3 = 1.20295
y(0.06) = y_4 = 1.31308
y(0.04) = y_5 = 1.43875
y(0.04) = y_6 = 1.58345

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